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On Finding Integer Solutions To Non-homogeneous Ternary Bi-quadratic Equation $2(x^2+y^2)-xy=57z^4$

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Abstract:

This paper concerns with the problem of obtaining non-zero distinct integer solutions to the non-homogeneous ternary bi-quadratic equation $2(x^2+y^2)-xy=57z^4$. Different sets of integer solutions are illustrated.

Keywords: Non-homogeneous bi-quadratic ,ternary bi-quadratic , integer solutions

I.INTRODUCTION:

The Diophantine equations are rich in variety and offer an unlimited field for research [1-4]. In particular refer [5-32] for a few problems on bi-quadratic equation with three unknowns. This paper concerns with yet another interesting bi-quadratic diophantine equation with three variables given by $2(x^2+y^2)-xy=57z^4$ for determining its infinitely many non-zero distinct integral solutions

II. METHOD OF ANALYSIS:

The non-homogeneous ternary bi-quadratic equation under consideration is

$$2(x^2+y^2)-xy=57z^4 \tag{1}$$

To start with, it is seen that (1) is satisfied by the integer triple

$$(x,y,z)=(5k^2,-k^2,k)$$

However ,there are other sets of integer solutions to (1) .Introduction of the linear transformations

$$x = u + 3v, y = u - 3v \tag{2}$$

in (1) leads to

$$u^2 + 15v^2 = 19z^4 \tag{3}$$

We solve (3) through different ways and using (2) obtain different sets of solutions to (1).

Way 1:

Let

$$z = a^2 + 15b^2 \tag{4}$$

Write 19 on the R.H.S. of (3) as

$$19 = (2 + i\sqrt{15})(2 - i\sqrt{15}) \tag{5}$$

Substituting (4) & (5) in (3) and employing the method of factorization, consider

$$u + i\sqrt{15}v = (2 + i\sqrt{15})(a + i\sqrt{15}b)^4 \tag{6}$$

On equating the real and imaginary parts in (6), and employing (2), the values of x, y

are given by

$$\left.\begin{aligned} x &= 5(a^4 - 90a^2b^2 + 225b^4) - 9(4a^3b - 60ab^3), \\ y &= -(a^4 - 90a^2b^2 + 225b^4) - 21(4a^3b - 60ab^3) \end{aligned}\right\} \tag{7}$$

Thus,(4) and (7) represent the integer solutions to (1).

Note 1:

The integer 19 on the R.H.S. of (3) is also represented as

$$19 = \frac{(17 + \mathrm{i}\sqrt{15})(17 - \mathrm{i}\sqrt{15})}{16}$$

Repetition of the above process leads to a different set of integer solutions to (1).

Way 2:

Rewrite (3) as

$$u^2 + 15v^2 = 19z^4 * 1 \tag{8}$$

Consider 1 on the R.H.S. of (8) as

$$1 = \frac{(1 + i\sqrt{15})(1 - i\sqrt{15})}{16} \tag{9}$$

Following the analysis similar to Way1, and replacing a by $2A$, b by $2B$,we get

the values of x, y, z satisfying (1) are given by

$$x(A,B) = -16(A^4 - 90A^2B^2 + 225B^4) - 336(4A^3B - 60AB^3),$$
$$y(A,B) = -88(A^4 - 90A^2B^2 + 225B^4) - 24(4A^3B - 60AB^3),$$
$$z(A,B) = 4(A^2 + 15B^2)$$

Note 2:

The integer 1 on the R.H.S. of (8) is also expressed as

$$1 = \frac{(15r^2 - s^2 + i2\sqrt{15}rs)(15r^2 - s^2 - i2\sqrt{15}rs)}{(15r^2 + s^2)^2},$$
$$1 = \frac{(11 + i3\sqrt{15})(11 - i3\sqrt{15})}{256},$$
$$1 = \frac{(61 + i5\sqrt{15})(61 - i5\sqrt{15})}{4096}$$

Repeating the above process, different sets of solutions to (1) are obtained.

Way 3:

Express (3) in the form of ratios as

$$\frac{u + 2z^2}{z^2 + v} = \frac{15(z^2 - v)}{u - 2z^2} = \frac{\alpha}{\beta}, \beta \neq 0 \tag{10}$$

Solving the above system of double equations (10),one has

$$x = -\alpha^2 + 42\alpha\beta + 15\beta^2, y = 5\alpha^2 + 18\alpha\beta - 75\beta^2, z^2 = \alpha^2 + 15\beta^2 \tag{11}$$

The third equation in (11) is satisfied by

$$\alpha = 15p^2 - q^2, \beta = 2pq \tag{12}$$

and

$$z = 15p^2 + q^2 \tag{13}$$

Substituting (12) in the first two equations of (11), the values of x, y are given by

$$\left.\begin{aligned} x &= -225p^4 - q^4 + 90p^2q^2 + 1260p^3q - 84pq^3, \\ y &= 1125p^4 + 5q^4 - 450p^2q^2 + 540p^3q - 36pq^3 \end{aligned}\right\} \tag{14}$$

Thus, (13) and (14) represent the integer solutions to (1).

Note 3:

One may also write (1) in the form of ratios as

$$\frac{u+2z^2}{z^2-v} = \frac{15(z^2+v)}{u-2z^2} = \frac{\alpha}{\beta}, \beta \neq 0,$$

$$\frac{u+2z^2}{15(z^2-v)} = \frac{(z^2+v)}{u-2z^2} = \frac{\alpha}{\beta}, \beta \neq 0,$$

$$\frac{u+2z^2}{15(z^2+v)} = \frac{(z^2-v)}{u-2z^2} = \frac{\alpha}{\beta}, \beta \neq 0$$

The repetition of the above process gives three more integer solutions to (1).

Way 4:

Taking

$$z^2 = X + 15T, v = X + 19T, u = 2U \tag{15}$$

in (3),it is written as

$$X^2 = 285T^2 + U^2 \tag{16}$$

Express (16) as the system of double equations as below in Table 1:

Table 1: System of double equations

System	I	II	III	IV

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$X+U$	285T	95T	57T	19T
$X-U$	T	3T	5T	15T

Solving each of the above system of equations ,the values of X, U, T are obtained. Then, from (15) and (2),the solutions to (1) are obtained .For brevity, the corresponding solutions are exhibited below:

Solutions from system I :

$$x = 770*158*a^2, y = -202*158*a^2, z = 158a$$

Solutions from system II :

$$x = 296*64*a^2, y = -112*64*a^2, z = 64a$$

Solutions from system III :

$$x = 202*46*a^2, y = -98*46*a^2, z = 46a$$

Solutions from system IV :

$$x = 112*32*a^2, y = -104*32*a^2, z = 32a$$

Note 4:

Apart from (15),one may also take

$$z^2 = X - 15T, v = X - 19T, u = 2U$$

giving four more sets of solutions to (1).

Way 5:

Treating (1) as a quadratic in x and solving for x , we have

$$x=\frac{Y\pm\sqrt{114z^4-15Y^2}}{2} \tag{17}$$

where

$$y=2Y \tag{18}$$

To eliminate the square-root on the R.H.S. of (17) , consider

$$\alpha^2+15Y^2=114z^4 \tag{19}$$

Write the integer 114 on the R.H.S. of (19) as

$$114=\frac{(21+i\sqrt{15})(21-i\sqrt{15})}{4} \tag{20}$$

Substituting (4) & (20) in (19) and employing the method of factorization , consider

$$\alpha+i\sqrt{15}Y=\frac{(21+i\sqrt{15})(a+i\sqrt{15}b)^4}{2}$$

On equating the real and imaginary parts , one has

$$\left.\begin{aligned}\alpha&=\frac{21(a^4-90a^2b^2+225b^4)-15(4a^3b-60ab^3)}{2},\\ Y&=\frac{(a^4-90a^2b^2+225b^4)+21(4a^3b-60ab^3)}{2}\end{aligned}\right\} \tag{21}$$

In view of (18) , one has

$$y=(a^4-90a^2b^2+225b^4)+21(4a^3b-60ab^3) \tag{22}$$

Using (21) in (17) , there are two representations for x given by

$$x=11(a^4-90a^2b^2+225b^4)+3(4a^3b-60ab^3) \tag{23}$$

$$x = -10(a^4 - 90a^2b^2 + 225b^4) + 18(4a^3b - 60ab^3)$$

(24)

Thus, (4) ,(22) , (23) and (4) ,(22) ,(24) give two sets of integer solutions to (1).

Note 5:

Rewrite (19) as

$$\alpha^2 + 15Y^2 = 114z^4 * 1 \tag{25}$$

Substituting (4) ,(9) ,(20) in (25) and employing the method of factorization ,

consider

$$\alpha + i\sqrt{15}Y = \frac{(21 + i\sqrt{15})\ (1 + i\sqrt{15})(a + i\sqrt{15}b)^4}{8}$$

Following the analysis as in Way 1, replacing a by $2A$ and b by $2B$ we get the following two sets of integer solutions to (1).

Set 1:

$$x(A,B) = 16A^4 - 1440A^2B^2 + 3600B^4 + 1344A^3B - 20160AB^3$$
$$y(A,B) = 88A^4 - 7920A^2B^2 + 19800B^4 + 96A^3B - 1440AB^3$$
$$z(A,B) = 4A^2 + 60B^2$$

Set 2:

$$x(A,B) = 28A^4 - 2520A^2B^2 + 6300B^4 - 1296A^3B + 19440AB^3$$
$$y(A,B) = 88A^4 - 7920A^2B^2 + 19800B^4 + 96A^3B - 1440AB^3$$
$$z(A,B) = 4A^2 + 60B^2$$

It is worth to note that,for integer solutions to (1),the values of a,b in the above two sets should be of the same parity.

III.CONCLUSION:

An attempt has been made to obtain non-zero distinct integer solutions to the non-homogeneous bi-quadratic diophantine equation with three unknowns given by

$2(x^2+y^2)-xy=57z^4$.One may search for other sets of integer solutions to the considered equation as well as other choices of the fourth degree diophantine equations with multi-variables

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